A REVIEW ON DEVELOPMENT OF MATHEMATICAL MODEL OF GROUND VEHICLE

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Abstract**—** In recent years, the autonomous ground vehicle has become a significant area of research due to its promising impacts on social and technical problems: a time-saving tool for its fast response time, a lifesaving technology for its automatic decisions in vulnerable conditions, and last but not the least a key to less chaotic traffic scenario. A literature review for the sake of a mathematical model for a passenger car is done in this paper by accumulating the basic law of physics, and the basic theories of mathematical modeling and system types are also discussed. Several established theorems and assumptions are gathered together in this paper, which can be further used for the development of a vehicle model with several degrees of freedom. It is observed from the review that the inclusion of road curvature, steering angle, side slip angle, lateral offset, and yaw rate within the dynamics of the vehicle will result in more reliable modeling of the vehicle. The information and literature of this paper can be used in the field of autonomous vehicle research.

Keywords**—** Autonomous vehicle, Ground vehicle Mathematical model, Vehicle dynamics, Dynamic Systems.

I. INTRODUCTION

The interest in the implementation of autonomy in the ground vehicle has been growing over the last few years rapidly, to reduce injuries and fatalities and also ease the driving situation [1]. Today an upper-range car is available with automatic handling responses the vehicle, such as Anti-Lock Braking (ABS), Traction Control (TCS), Electronic Stability Control (ESC), adaptive cruise control (ACC), and Roll Stability Control (RSC), lane departure warning systems (LDWS), lane-keeping systems (LKS) and yaw stability control systems [2], shown in figure 1.

The proper way to deal with any problem involving time-dependent behavior requires a mathematical description of the object or process involved, this description is widely known as the mathematical model of that specific system. Usually, the model of a dynamic system is a set of differential equations. A mechanical system can be represented either by a kinematic or dynamic mathematical model. The kinematic model represents features of the system without considering the

Fig. 1: Features in Autonomous cars [3]

causes of motion, whereas the dynamic model represents features of the system that are related to motion and force [4]. Depending on the number of independent spatial coordinates required to define a system, the degrees of freedom can be selected.

The main objective of this paper is to overview the existing mathematical modeling of a ground vehicle. The number of degrees of freedom included by researchers in their work is also mentioned in this paper. Moreover, the various dynamic variables of the ground vehicle, such as yaw rate, steering angle, road curvature, angular displacement, lateral displacement, side slip angle, and the advantages of these additives are also explained. Again, the vehicle dynamics, representation of mathematical models of ground vehicles, and dynamic system types of systems are also discussed in this paper for a better understanding of mathematical modeling

II. BACKGROUND OF MATHEMATICAL MODELS OF DYNAMIC SYSTEMS

The heuristic approach is to model a system based on its dynamics, as this accounts for the forces and torques

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acting on the system. Depending on the forces and torques produced from the tires, gravity, and aerodynamics [4], two types of forces and torques acting on a vehicle, lateral and longitudinal. Hence, there are two types of dynamics of a vehicle, lateral dynamics and longitudinal dynamics [5].

The choice of dynamics to model a vehicle depends on the objective of the system. For example, an obstacle detection system mostly depends on the control of the longitudinal velocity of the vehicle, so in the case of obstacle detection, one should use the longitudinal dynamics of the system [6]. On the other hand, if the control objective is to steer the vehicle in a manner so that it remains within a predefined path, which is lane-keeping, in that case, lateral dynamics of the vehicle should be used for vehicle modeling [7]. In this paper, the developed theories are presented which are needed for developing a model with lateral dynamics of the vehicle.

Mathematical models of vehicles are significant in the area of control of automotive vehicles. Several forms of mathematical modeling of the ground vehicle are available for simulation studies with different levels of complexity and accuracy [8] [4] [9]. The kinematics and dynamics of ground vehicles have been developed and redeveloped since the 19th century [10]. The dynamics of a moving mechanical system are dependent on its movement on the road surface and the forces exerted on it by the tires, gravity, and aerodynamics. A suitable study on the vehicle parameters, components, and parameters dependent on the interaction with the road, can result in satisfactory mathematical modeling of the vehicle [11].

Despite many forms of vehicle mathematical modeling, still there is scope to improve the modeling of vehicles by including more significant parameters. The choice of model should depend on the desired control objectives and desired autonomous features. The control of longitudinal dynamics and control of lateral dynamics results in adaptive cruise control and lane departure warning system respectively. Moreover, the control of a combination of lateral and longitudinal dynamics can result in a fully automated car. Since the focus of this research is on the lane-keeping system of ground vehicles, so the lateral dynamics are our prime concern. Many researchers have developed suitable lateral dynamics for the ground vehicle so far.

A five-state mathematical model of a bus-like vehicle is used by the authors of [12]. They have used side-slip angle, yaw rate, the angle between the centerline of the vehicle and tangent to the reference, steering angle of the front wheel, and lateral position as the states of the vehicle. The curvature of the road is a reference input to the model input, moreover, a wind force is also included as an input. Later, a four-order vehicle model is used where the control input is the steering angle and road curvature is a disturbance input [13]. The longitudinal velocity, yaw rate, lateral position, and angular position deviation are four states of the vehicle. Again, the authors of [14] develop a four-order vehicle kinematic modeling used for lateral control. The lateral position, longitudinal position, angular position of the car, and the steering wheel angle concerning the car's longitudinal axis are used as vehicle states.

All the necessary parameters like error variable, yaw rate, steering, road curvature, road adherence coefficient, and sideslip were taken into account in the paper [15]. Mathematical modeling of the ground vehicle has been done based on lateral and angular error variables from the road centerline to the center of gravity of the vehicle [8]. In another paper control objective includes both controls of lateral and longitudinal dynamics, so the vehicle states are a combination of both dynamics, a vehicle model is used along with a driver model [16]. The input is the conjunction of controller and driver inputs, which are steering angle and braking ratio. They have used road curvature as a reference input. For lane-keeping controller design, input delay was introduced in the vehicle mathematical modeling to make it more similar to the human driver model, as the human brain takes time to act on the change of road curvature before it steers [17].

III. REPRESENTATION OF MATHEMATICAL MODELS OF DYNAMIC SYSTEMS

There are two domains for the synthesis and analysis of control systems, frequency and time [18]. The differential equation established by mathematical modeling is converted either in the frequency domain or in the time domain, hence in transfer function or state-space form. The conventional method of representing a dynamic system is to use the transfer function, defined below,

A. *Definition 1* [8]

The transfer function of a linear, time-invariant, differential equation system is defined as the ratio of the Laplace transform of the output /(response function/) to the Laplace transform of the input /(driving function/) under the assumption that all initial conditions are zero. Some of the disadvantages of the conventional method, tends the researcher has to represent the system in state-space form, for example, a transfer function is limited to the only linear time-invariant system, it does not represent the physical nature of the system, initial conditions are ignored [19]. Introduction time-domain

main representation of dynamic systems has overcome the limitations of the transfer function. By the definition given below, the state-space system can be explained.

B. Definition 2 [8]

The n-dimensional space whose coordinate axes consist of the x_1 axis, x_2 axis, ... x_n axis, where x_1, x_2, ..., x_n are state variables, is called state space. Any state can be represented by a point in the state space. The differential equation of a system can be expressed in a compact form [20], such as,

$$\dot{x}(t) = f(x,u,t) \quad (1)$$
$$y(t) = g(x,u,t) \quad (2)$$

This set of equations is a state equation and output equation respectively. Again, if the vector function f and g involve time in an explicit manner, then the system is a time-varying system. The equations (1) and (2) can be linearized about the operating point of the system state. Eventually, the linearized form of state equation and output equation can be written as,

$$\dot{x}(t) = A(t)\hat{x}(t) + B(t)\hat{u}(t) \quad (3)$$
$$y(t) = C(t)\hat{x}(t) + D(t)\hat{u}(t) \quad (4)$$

The where, $t \in R$ denotes time, $\hat{x}(t) \in R^{n1}$,is state vector, $\hat{y}$ (t) ∈ R^{P1} is output vector, $\hat{x}(t) \in R^{m1}$ is the input vector, $A(t) \in R^{n1*n1}$ is the dynamics matrix, $B(t) \in R^{n1*m1}$, $C(t) \in R^{p1*x1}$ is output matrix, $D(t) \in R^{p1*m1}$ is the feedthrough matrix. Because the system to be dealt with here is a linear time-invariant system so, the constant matrices A, B, C, D are constants and do not depend on the time, t. Hence, the state-space representation of a linear time-invariant system is,

$$\dot{x}(t) = A\hat{x}(t) + B\hat{u}(t) \quad (5)$$
$$y(t) = C\hat{x}(t) + D\hat{u}(t) \quad (6)$$

The system that can change its behavior in response to unanticipated events during operation is called an autonomous system. Mathematically, autonomous system can be explained by the following theorem,

C. Definition 2 [8]

If the system does not depend explicitly on time, that is, $f(x,t) = f(x)$, it is said to be autonomous. Moreover, there are some unique advantages of a linear time-invariant system. A detailed classification of the dynamic system is shown in figure 2. From this figure,

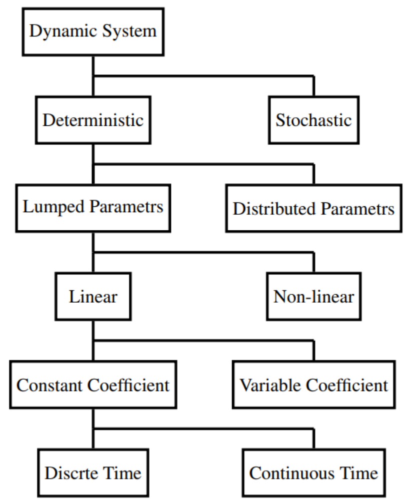

Fig. 2: Classification of dynamic systems [21]

it is observed that dynamic systems are divided into two classes, deterministic and stochastic systems. Later the deterministic system is divided into lumped and distributed parameter systems. Afterward based on linearity the lumped parameter systems again can be divided into linear and non-linear systems. According to the type of the coefficients of the system, the linear system can be classified into constant or variable types. Lastly, according to the domain of the system the linear time-invariant systems can be divided into discrete and continuous-time systems. Depending on the number of independent spatial coordinates required to define a system, the degrees of freedom can select.

Linear systems are ideal systems that satisfy the superposition principle [22]. That is why, among many advantages of linear systems, one is, that the impulse response or the step response is enough to predict the output behavior of the system. Moreover, Linear systems, on the other hand, are well understood, and there is vast literature on the analysis of linear systems. The simplicity of linear control theory is in comparison to the mathematics of non-linear control theory, evidence of more of the classical mathematics and physics [23]. It is therefore appealing to try to approximate the non-linear dynamics system into linear systems. Again, the output of a linear MIMO (multi-input multi-output) system can be analyzed by decomposing the system. Finally, the linear system depends only on the current input values rather than past inputs, so it is easy to predict the nature of a linear system.

IV. MATHEMATICAL MODELING OF GROUND VEHICLE

The mathematical modeling of the ground vehicle has been expansively studied in the past years. Several vehicle models can be found in rich literature[24] [25]

[4]. There are two different approaches for modeling mechanical systems - kinematic and dynamic [7]. Being relatively simple, the kinematic model is very popular for research. However, the dynamic model is more accurate for real-world applications as it concerns the effects of forces on the motion of bodies with mass and rotational inertia. At high speed, the velocity at each wheel is no longer in the direction of the wheel. For reliability, at high-speed conditions, the dynamic model of the vehicle should be used for academic research. This is why most of the ground vehicle control research is based on the dynamics model or its variations since it represents a more realistic approach. In this section, the dynamic mathematical model used throughout this research is formulated. In many previous works, dynamic mathematical modeling of ground vehicles has already been developed. Before attempting the mathematical modeling of the vehicle, it is needed to study the vehicle parameters and components vigorously, to determine the type of forces that will be produced on the vehicle at particular maneuver and trim conditions. Additionally, it is also necessary to know how the vehicle will respond to the developed forces. That is why it is essential to establish a rigorous approach to modeling the vehicle and the conventions that will be used to describe motion.

In the research work of [12], a five-degree of freedom mathematical modeling was used for automatic steering control but the simulation results were validated only for the bus-like ground vehicle rather than the car-like ground vehicle. Again, a four-state vehicle model was developed for controlling a ground vehicle [26]. The road curvature is a disturbance input in the model but the side-slip angle and the road tire friction coefficient were not included in the modeling. If only the kinematics of the vehicle is taken into account for mathematical modeling, the model ignores the dynamics of the vehicle, which is not very realistic [27].

Side-slip angle, yaw rate, angular displacement offset, and lateral displacement offset are used as the vehicle state in the research presented in [15]. However, the lateral displacement offset does not depend on the change of road curvature but rather only on the look-ahead distance. Therefore, this model may be good for only straight-line roads but not very realistic for roads with different radii. Again, in the work of [24], a control method of lateral dynamics of the ground vehicle was presented, in this paper, the road radius was not taken into account. Besides yaw rate, the side-slip angle was not included in the equations of motion of the vehicle. A combination of lateral and longitudinal dynamics is used, but the road curvature is not treated as a disturbance input [28]. The authors of [16] did not include side slip angle, and yaw rate as the state of the vehicle, moreover, there was no disturbance included in the model.

V. Conclusion

The major contribution of this paper is, that the theories behind the development of a mathematical modeling ground vehicle are accumulated, which could be very aidful to establish a model of a passenger car with lateral dynamics. The knowledge of the dynamics could lead to a realistic model of ground vehicle, hence support to the growing technology of the autonomous car. The future path is to develop a required degree of freedom ground vehicle model with the assumptions and established models.

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